

On maximal globally hyperbolic vacuum space-times*

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Abstract

We prove existence and uniqueness of maximal global hyperbolic developments of vacuum general relativistic initial data sets with initial data (g, K) in Sobolev spaces $H^s \oplus H^{s-1}$, $\mathbb{N} \ni s > n/2 + 1$.

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1 Introduction

The celebrated Choquet-Bruhat – Geroch theorem [5] asserts that to every smooth vacuum general relativistic initial data set (\mathcal{S}, g, K) one can associate a unique, up to isometries, smooth solution of the vacuum Einstein equations. This should be compared to the local existence theory, where solutions with Sobolev initial data $(\gamma, K) \in H^s \oplus H^{s-1}$ are constructed for $s > n/2 + 1$. The aim of this work is to make a step towards bridging this gap, and to prove:

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THEOREM 1.1 *Consider a vacuum Cauchy data set (\mathcal{S}, γ, K) , where \mathcal{S} is an n -dimensional manifold, $\gamma \in H_{\text{loc}}^s(\mathcal{S})$ is a Riemannian metric on \mathcal{S} , and $K \in H_{\text{loc}}^{s-1}(\mathcal{S})$ is a symmetric two-tensor on \mathcal{S} , satisfying the general relativistic vacuum constraint equations, where $\mathbb{N} \ni s > n/2 + 1$. Then there exists a unique up to isometries vacuum space-time (\mathcal{M}, g) , called the maximal globally hyperbolic vacuum development of (\mathcal{S}, γ, K) , with an embedding $i: \mathcal{S} \rightarrow \mathcal{M}$ such that $i^*g = \gamma$, and such that K corresponds to the extrinsic curvature tensor of $i(\mathcal{S})$ in \mathcal{M} . (\mathcal{M}, g) is inextendible in the class of globally hyperbolic space-times with a vacuum metric.*

To avoid ambiguities, global hyperbolicity here is the requirement that every inextendible causal curve meets $i(\mathcal{S})$ precisely once.

There is little doubt that the condition $\mathbb{N} \ni s > n/2$ can be relaxed to $\mathbb{R} \ni s > n/2$ using paradifferential techniques, see e.g. [2, 25]. We reduce this question to the problem of verifying conditions H1-A3, p. 10, compare Theorem 3.1 below. It is conceivable that the result generalises to more general classes of initial data for which local existence and uniqueness of solutions holds, such as e.g. those considered in [20–22, 26, 27], but this remains to be seen.

The proof here is an adaptation of that in [7], using the Planchon-Rodnianski uniqueness argument [27], an extension of the analysis in [4, Appendix A] to manifolds of $H_{\text{space,loc}}^{s+1}$ differentiability class, and the causality theory for continuous metrics in [11].

It might be useful to comment upon the differentiability thresholds that arise in previous proofs of the theorem. First, all the proofs use various elements of causality theory which have only been consistently developed using standard approaches for smooth, or C^2 [9] metrics. So, without further detailed justifications, that part of the proof that appeals to causality theory would require at least C^2 differentiability of the metric. The original proof in [5] assumes explicitly smoothness at the outset, and invokes existence and uniqueness of geodesics, which fails for metrics which are not $C^{1,1}$. Similarly geodesics are invoked in the proofs given in [16, 28]. The sketchy argument presented in [6] is the only one that does not explicitly use geodesics, but the authors do not spell out the differentiability of the metric they had in mind for their proof.

The argument in [7] (which proves a more general result, with the Choquet-Bruhat – Geroch theorem being a straightforward consequence of Proposition 2.2 there) was presented for smooth metrics because neither the low-differentiability causality theory, nor the Planchon-Rodnianski uniqueness argument [27] were available at the time. However, the proof was written using arguments which generalise to metrics with Sobolev differentiability if the associated causality theory goes through. Inspection of [7] shows that the elements of causality theory needed there arise in the proof of Lemma 2.3 of that reference, and are 1) existence and causality of accumulation curves; 2) the fact that a causal curve which is not null everywhere can be deformed to a timelike curve at end points fixed; and 3) some further technical issues related to causal properties of $\partial\mathcal{D}_J$. It is not obvious, but proved in [11] (see also [14]), that point 1) remains true for continuous metrics, but that point 2) is wrong for continuous metrics; then the usual arguments addressing 3) fail. This part of the argument is replaced by the rather more involved argument starting after the proof of Lemma 3.5, p. 12 below.

2 Existence of maximal developments

As a first step in the proof of the Choquet-Bruhat – Geroch theorem, one constructs space-times which are maximal with respect to a set of properties. This begs the question, if and when is such a construction possible. We start by addressing this. Some notation is in order.

Let W denote a set of properties of a manifold, possibly equipped with some supplementary structure such as a metric. Here all manifolds are connected, paracompact, Hausdorff, of at least C^1 differentiability class. When talking about space-time, the dimension will be denoted by $n + 1$. Thus, spacelike hypersurfaces, or their models, will be of dimension n . The property W will include differentiability requirements, *e.g.* $C^{k,\alpha}$, or analyticity, or some Sobolev class, and it might, or might-not, include some further requirements.

A manifold will be said to be Lorentzian if it is equipped with a metric tensor, perhaps defined only almost everywhere, of a differentiability class adapted to that of W . For example, a natural class W could be manifolds with a $C^{k,\alpha}$ atlas, $k \geq 1$, and metrics of $C^{k-1,\alpha}$ differentiability class. It is useful to keep in mind that W can denote a rather complicated structure. For the purpose of the Cauchy problem in general relativity we will be using a $H_{\text{space,loc}}^s$ structure, defined as follows:

DEFINITION 2.1 *Let $s \in \mathbb{R}$. A Lorentzian manifold (\mathcal{M}, g) will be said to be of $H_{\text{space,loc}}^s$ differentiability class if every point $p \in \mathcal{M}$ has a coordinate neighborhood $\mathcal{U}_p = I \times \mathcal{V}_p$, where I is the range of a time coordinate $t \equiv x^0$, with the following properties: On every level set \mathcal{S}_τ of t the metric components $g_{\mu\nu}$ are of H^s differentiability class, and their time-derivatives of order $0 \leq k \leq \lfloor s \rfloor$ are of H^{s-k} differentiability. Furthermore the functions*

$$I \ni t \mapsto \|\partial_0^k g_{\mu\nu}|_{\mathcal{S}_t}\|_{H^{s-k}}, \quad 0 \leq k \leq \lfloor s \rfloor,$$

are continuous.

Thus, the index “space” in $H_{\text{space,loc}}^s$ denotes the fact that the differentiability of the metric is defined in terms of Sobolev spaces on spacelike hypersurfaces. The “loc” index, shorthand for “local”, refers to the fact that the relevant H^s norms are finite on every compact set; note that the corresponding integrals are not necessarily finite when calculated over sets with non-compact closure.

One expects that the maximal atlas compatible with a $H_{\text{space,loc}}^s$ structure will consist of maps which are of differentiability class $H_{\text{space,loc}}^{s+1}$. This fact, which is one of the elements of the proof of Theorem 1.1, is established in Proposition 3.7 below for $s \in \mathbb{N}$, $s > n/1 + 1$.

Since all our manifolds are assumed to be C^1 , maps between them will also be C^1 in any case, unless explicitly indicated otherwise.

A Lorentzian manifold will be called vacuum if the equations $R_{\mu\nu} = 0$ can be defined, perhaps in a distributional sense, and if $R_{\mu\nu} = 0$ holds. Note that the Christoffel symbols can be defined for metrics with $g^{\mu\nu} \in L_{\text{loc}}^\infty$ and which have distributional derivatives in L_{loc}^1 . The equation $R_{\mu\nu} = 0$ can be defined in a distributional sense if moreover the distributional derivatives of the metric are in L_{loc}^2 .

The standard theory of PDEs constructs $H_{\text{space,loc}}^s$ solutions of the Einstein equations with $\mathbb{R} \in s > n/2 + 1$, with an atlas in which the coordinate functions are harmonic [18, 27] (compare [10, Section 4.3]).

The more recent work in [21] together with [24, Theorem 7.1] constructs vacuum metrics in dimension $3 + 1$ with $s > 2$, assuming asymptotically flat initial data on \mathbb{R}^3 with $\text{tr}K = 0$.

Since we will be solving the Cauchy problem, we will need to consider an embedding i of a spacelike hypersurface \mathcal{S} into (\mathcal{M}, g) . The embedding should be compatible with the structures available; we will say that i is *W-compatible* when this is the case. For example, for $C^{k,\alpha}$, or smooth, or analytic, manifolds it would be natural to consider maps which are also of $C^{k,\alpha}$ class, or smooth, or analytic. So, in this case, a *W-compatible* embedding would be required to be $C^{k,\alpha}$, or smooth, etc. For $H_{\text{space,loc}}^s$ manifolds it is natural to consider embeddings $i : \mathcal{S} \rightarrow \mathcal{M}$ such that the pull-backs (γ, K) of the metric and of the extrinsic curvature from $i(\mathcal{S})$ to \mathcal{S} are in $H_{\text{loc}}^s \otimes H_{\text{loc}}^{s-1}$; this is our definition of *W-compatible* embedding for $H_{\text{space,loc}}^s$ manifolds. The resulting hypersurfaces $i(\mathcal{S})$ will be called *W-compatible*.

To make things clear, the property *W* of main interest in this work is: (\mathcal{M}, g) is a Hausdorff, paracompact, connected globally hyperbolic vacuum C^1 manifold of $H_{\text{space,loc}}^s$ differentiability class.

Nevertheless, a reader only interested in smooth vacuum space-times can assume that *W* is the property that (\mathcal{M}, g) is a smooth, Hausdorff, paracompact, connected globally hyperbolic vacuum manifold with a smooth metric. A *W-compatible* embedding i means then that i is smooth, and a *W-compatible* submanifold means a smooth submanifold. Similarly for $C^{k,\alpha}$ or analytic manifolds.

However, the next lemma works with any notions of *W*-manifold and *W-compatible* embedding which can be formulated within the framework of set theory as described e.g. in [19, Appendix]:

THEOREM 2.2 *Let \mathcal{S} be a n -dimensional manifold and let (\mathcal{M}, g, i) be a Lorentzian $(n + 1)$ -dimensional *W*-manifold (\mathcal{M}, g) with a *W-compatible* embedding $i : \mathcal{S} \rightarrow \mathcal{M}$. Suppose that the property *W* implies that*

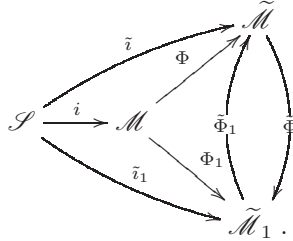
$$\begin{aligned} & \text{the only isometry of } (\mathcal{M}, g) \text{ which is the identity} \\ & \text{on a } W\text{-compatible hypersurface is the identity map.} \end{aligned} \quad (2.1)$$

*Then there exists a Lorentzian *W*-manifold $(\tilde{\mathcal{M}}, \tilde{g}, \tilde{i})$ with a *W-compatible* embedding $\tilde{i} : \mathcal{S} \rightarrow \tilde{\mathcal{M}}$ and a C^1 isometric embedding $\Phi : \mathcal{M} \rightarrow \tilde{\mathcal{M}}$ satisfying $\tilde{i} = \Phi \circ i$ such that $\tilde{\mathcal{M}}$ is inextendible in the class of Lorentzian *W*-manifolds with a *W-compatible* embedding of \mathcal{S} .*

- REMARKS 2.3**
1. The C^1 differentiability threshold for \mathcal{S} and \mathcal{M} cannot be weakened in the proof below. The author ignores whether or not the C^1 differentiability is necessary.
 2. One expects the differentiability of Φ to be determined by that of the metric. For example Φ will be $C^{k+1,\alpha}$ if the metric is $C^{k,\alpha}$; smooth or analytic if the metric is, etc. This is proved by a bootstrap argument applied to (2.4) below. See [8, Appendix A] for the analytic case.
 3. The maximal manifolds $(\tilde{\mathcal{M}}, \tilde{g})$ need not be unique, and may depend upon *W*. A non-trivial example of *W* dependence, with $W = C^{k,\alpha}$, is given by a class of Robinson–Trautman (RT) space-times studied in [12], which for $k + \alpha \geq 123$ do not admit any non-trivial future extensions, while for $k + \alpha < 118$ possess an infinite number of non-isometric vacuum RT extensions.

PROOF: For $\ell \geq n$ let A_ℓ denote the set¹ of subsets of \mathbb{R}^ℓ which are n -dimensional manifolds, set $A_\infty = \cup_{\ell=0}^\infty A_\ell$. By a famous theorem of Whitney [29] every $(C^1, \text{connected, paracompact, Hausdorff})$ manifold can be embedded in \mathbb{R}^ℓ for some ℓ , which shows that every manifold has a representative which is an element of A_∞ . It follows that without loss of generality a manifold can be viewed as an element of A_∞ , and we shall do so. With this definition the collection of all C^1 manifolds is A_∞ , and therefore is a set. It follows from the axioms of set theory that the collection of all C^1 manifolds which are W manifolds forms a set. Now, a Lorentzian manifold can be identified with a subset of the bundle $T_2\mathcal{M}$, where $T_2\mathcal{M}$ is the bundle of 2-covariant tensors on \mathcal{M} . Next, a map i from \mathcal{S} to \mathcal{M} can be identified with a subset of the product $\mathcal{S} \times \mathcal{M}$. One easily concludes that the collection $\mathcal{M}_{W,\mathcal{S}}$ of Lorentzian W -manifolds with a W -compatible embedding of \mathcal{S} forms a set.

Let (\mathcal{M}, g) be a Lorentzian W manifold with W -compatible embedding $i : \mathcal{S} \rightarrow \mathcal{M}$. Consider the subset $\mathcal{M}_W(M, g, i)$ of $\mathcal{M}_{W,\mathcal{S}}$ defined as the set of those Lorentzian manifolds $(\tilde{\mathcal{M}}, \tilde{g}, \tilde{i})$ with embedding of \mathcal{S} for which there exists an isometric C^1 embedding $\Phi : \mathcal{M} \rightarrow \tilde{\mathcal{M}}$ with a W -compatible embedding $\tilde{i} = \Phi \circ i$. On $\mathcal{M}_W(M, g, i)$ we can define a relation \prec as follows: $(\tilde{M}, \tilde{g}, \tilde{i}) \prec (\tilde{M}_1, \tilde{g}_1, \tilde{i}_1)$ if there exists an isometric C^1 embedding $\tilde{\Phi} : \tilde{\mathcal{M}} \rightarrow \tilde{\mathcal{M}}_1$ satisfying $\tilde{\Phi} \circ \tilde{i} = \tilde{i}_1$. We claim that \prec is a partial order; here the only non-obvious property is antisymmetry, namely if $(\tilde{M}, \tilde{g}, \tilde{i}) \prec (\tilde{M}_1, \tilde{g}_1, \tilde{i}_1)$ and $(\tilde{M}_1, \tilde{g}_1, \tilde{i}_1) \prec (\tilde{M}, \tilde{g}, \tilde{i})$, then $(\tilde{M}, \tilde{g}, \tilde{i}) = (\tilde{M}_1, \tilde{g}_1, \tilde{i}_1)$. So let $\tilde{\Phi} : \tilde{\mathcal{M}} \rightarrow \tilde{\mathcal{M}}_1$ and $\tilde{\Phi}_1 : \tilde{\mathcal{M}}_1 \rightarrow \tilde{\mathcal{M}}$ be the relevant embeddings:



Then $\tilde{\Phi}_1 \circ \tilde{\Phi}$ is an isometry of $(\tilde{\mathcal{M}}, \tilde{g})$ which is the identity on $\tilde{i}(\mathcal{S})$. By (2.1) the map $\tilde{\Phi}_1 \circ \tilde{\Phi}$ is the identity on $\tilde{\mathcal{M}}$, thus $\tilde{\Phi} \circ \tilde{\Phi}_1$ is the identity on $\tilde{\mathcal{M}}_1$ as well, proving that $(\tilde{M}, \tilde{g}, \tilde{i}) = (\tilde{M}_1, \tilde{g}_1, \tilde{i}_1)$ up to isometry, as desired.

If $A \subset \mathcal{M}_W(M, g, i)$ is a chain, define

$$\bar{\mathcal{M}} = \left(\cup_{(\tilde{M}, \tilde{g}, \tilde{i}) \in A} \tilde{\mathcal{M}} \right) / \sim,$$

where for $p \in \tilde{\mathcal{M}}$ and $q \in \tilde{\mathcal{M}}_1$ we set $p \sim q$ iff $q = \Phi(p)$, where $\Phi : \tilde{\mathcal{M}} \rightarrow \tilde{\mathcal{M}}_1$ is the isometric C^1 embedding such that $\Phi \circ \tilde{i} = \tilde{i}_1$. It is not too difficult to show that $\bar{\mathcal{M}}$ is a W manifold (Hausdorff, paracompact, connected), and a Lorentzian metric \bar{g} can be defined on $\bar{\mathcal{M}}$ in the obvious way. Since every $\tilde{\mathcal{M}}$ such that $(\tilde{M}, \tilde{g}, \tilde{i}) \in A$ can be embedded in $\bar{\mathcal{M}}$ as

$$\tilde{\mathcal{M}} \ni p \mapsto [p]_{\sim} \in \bar{\mathcal{M}},$$

it follows that $\bar{\mathcal{M}}$ is an upper bound for A . The Kuratowski-Zorn Lemma (cf. e.g. [19]) shows that $\mathcal{M}_W(M, g, i)$ has maximal elements, which had to be established. \square

¹See, e.g., [19][Appendix] for an overview of axiomatic set theory.

Before continuing, it appears useful to exhibit classes of space-times in which condition (2.1) is satisfied. The simplest case is that $C^{k,\alpha}$ manifolds, where $k + \alpha \geq 3$, with $C^{k-1,\alpha}$ metrics, and with C^1 submanifolds and embeddings:

PROPOSITION 2.4 *Let (\mathcal{M}, g) be a $C^{2,1}$, connected Lorentzian manifold with a $C^{1,1}$ metric, let $\Psi : \mathcal{M} \rightarrow \mathcal{M}$ be a C^1 map such that*

$$\Psi^*g = g, \quad \Psi|_S = id \quad (S \neq \emptyset),$$

where S is either

1. an open set, or
2. $S = \{p\}$ is a point $p \in \mathcal{M}$, in which case we further assume that $\Psi^*(p)$ is the identity, or
3. a C^1 submanifold of codimension 1, in which case we moreover assume that Ψ preserves time-orientation.

Then

$$\Psi = id.$$

REMARK 2.5 Note that each of the conditions is necessary, and that in point 1 and 3 neither size nor completeness requirements are imposed on S .

PROOF: Suppose first that S is an open set, let \tilde{S} be the largest open set such that $\Psi|_{\tilde{S}} = id$. Suppose that \tilde{S} is not closed, thus there exists $p \in \partial\tilde{S}$, let \mathcal{O} be any neighbourhood of p with a local coordinate system such that $x^\mu(p) = 0$, continuous differentiability of Ψ implies, in local coordinates,

$$\Psi^\mu(0) = 0, \quad \frac{\partial \Psi^\mu}{\partial x^\nu}(0) = \delta_\nu^\mu. \quad (2.2)$$

From $\Psi^*g = g$ one has

$$g_{\alpha\beta}(x) = g_{\mu\nu}(\Psi(x)) \frac{\partial \Psi^\mu}{\partial x^\alpha} \frac{\partial \Psi^\nu}{\partial x^\beta}, \quad (2.3)$$

$$\frac{\partial^2 \Psi^\mu}{\partial x^\alpha \partial x^\beta} = \Gamma_{\alpha\beta}^\sigma(x) \frac{\partial \Psi^\mu}{\partial x^\sigma} - \Gamma_{\nu\rho}^\mu(\Psi(x)) \frac{\partial \Psi^\nu}{\partial x^\alpha} \frac{\partial \Psi^\rho}{\partial x^\beta}, \quad (2.4)$$

where Γ denotes the Christoffel symbols of the metric g . Indeed, recall that (2.4) is obtained by differentiating (2.3) and algebraic manipulations when Ψ is C^2 . When Ψ is assumed to be C^1 only, the same manipulations shows that (2.4) holds in a distributional sense. But since the right-hand side is continuous, we conclude that Ψ is C^2 in any case.

Setting $A_\beta^\alpha \equiv \frac{\partial \Psi^\alpha}{\partial x^\beta}$, from (2.4) one obtains the following system of ODE's along rays emanating from the origin:

$$\begin{aligned} \frac{d\Psi^\mu}{dr} &= A_\beta^\mu x^\beta / r, \quad r = \left(\sum (x^\alpha)^2 \right)^{1/2}, \\ \frac{dA_\beta^\mu}{dr} &= \left(\Gamma_{\alpha\beta}^\sigma(x) A_\sigma^\mu - \Gamma_{\nu\rho}^\mu(\Psi(x)) A_\alpha^\nu A_\beta^\rho \right) x^\alpha / r. \end{aligned}$$

The initial conditions (2.2) together with uniqueness of solutions of systems of ODE's imply $\Psi^\mu = x^\mu$ in \mathcal{O} , which leads to a contradiction, and shows that $\partial\tilde{S} = \emptyset$, thus $\tilde{S} = \mathcal{M}$. This proves point 1.

Note that we have also shown that if $\Psi(p) = p$ and $\Psi^*(p) = \text{Id}$, then $\Psi = \text{Id}$ on a neighborhood \mathcal{O} of p , hence $\Psi = \text{Id}$ by point 1, and point 2 is proved as well.

Suppose now that S is a hypersurface, let $p \in S$. Then Ψ^* is the identity on $T_p S$ and preserves orientation. Elementary algebra shows that $\Psi^*(p)$ is the identity: Indeed, this is straightforward if $T_p S$ is spacelike or timelike. If S is null, let n, ℓ, e_A , $A = 2, \dots, n$, be a basis of $T_p S$ such that n and ℓ are null, the e_A 's are ON and orthogonal to ℓ and n , with ℓ and e_A tangent to S . Then Ψ^* is a Lorentz transformation that leaves invariant both ℓ and the space spanned by ℓ and n , and preserves orientation, hence is the identity. The result follows now by point 2. \square

We have the following ‘‘Lipschitz-harmonic’’ version of Proposition 2.4:

PROPOSITION 2.6 *Let (\mathcal{M}, g) be a globally hyperbolic connected Lorentzian $(n+1)$ -dimensional manifold with differentiable spacelike Cauchy surface \mathcal{S} . Let $\Psi : \mathcal{M} \rightarrow \mathcal{M}$ be a time-orientation preserving C^1 map such that*

$$\Psi^* g = g, \quad \Psi|_{\mathcal{S}} = \text{id},$$

If \mathcal{M} can be covered by wave-coordinates patches in which the metric is $C^{0,1}$, then

$$\Psi = \text{id}.$$

REMARK 2.7 We have chosen the wave-coordinate condition for simplicity. The argument applies to any systems of coordinates in which $\square_g x^\mu = F^\mu(x, g)$ with Lipschitz functions F^μ .

PROOF: Equation (2.4), understood distributionally, in coordinates where g is Lipschitz, shows that Ψ is $C^{1,1}$.

Let $p \in \mathcal{S}$, since Ψ_* is an isometry and leaves $T_p \mathcal{S}$ invariant, it preserves $(T_p \mathcal{S})^\perp$. As Ψ preserves time-orientation, Ψ_* maps the unit normal to \mathcal{S} to itself. It follows that Ψ_* is the identity at p ; in local coordinates, $\partial \Psi^\mu / \partial x^\nu|_{\mathcal{S}} = \delta^\mu_\nu$.

Let \mathcal{O} denote the domain of definition of some wave-coordinates in which the metric is locally Lipschitz, thus

$$0 = \square_g x^\mu = -g^{\alpha\beta} \Gamma_{\alpha\beta}^\mu. \quad (2.5)$$

Let $\mathcal{S}_{\mathcal{O};\tau} \subset \mathcal{O}$ denote the level set within \mathcal{O} of a differentiable time function t :

$$\mathcal{S}_{\mathcal{O};\tau} := \{t = \tau\} \cap \mathcal{O}.$$

Note that we are *not* assuming that $t = x^0$.

Consider a point x with coordinates x^μ such that $\Psi(x) \in \mathcal{O}$. Contracting (2.4) with the inverse metric, and using the wave-coordinates condition, one obtains

$$\begin{aligned} \square_g \Psi^\mu(x) &= g^{\alpha\beta}(x) \left(\frac{\partial^2 \Psi^\mu}{\partial x^\alpha \partial x^\beta} - \Gamma_{\alpha\beta}^\sigma(x) \frac{\partial \Psi^\mu}{\partial x^\sigma} \right) \\ &= -g^{\alpha\beta}(x) \Gamma_{\nu\rho}^\mu(\Psi(x)) \frac{\partial \Psi^\nu}{\partial x^\alpha} \frac{\partial \Psi^\rho}{\partial x^\beta}. \end{aligned} \quad (2.6)$$

Setting $\psi^\mu := \Psi^\mu - x^\mu$, this can be rewritten in the form

$$\begin{aligned}
\Box_g \psi^\mu(x) &= -g^{\alpha\beta}(x) \Gamma_{\nu\rho}^\mu(x + \psi(x)) \left(\frac{\partial \psi^\nu}{\partial x^\alpha} + \delta_\alpha^\nu \right) \left(\frac{\partial \psi^\rho}{\partial x^\beta} + \delta_\beta^\rho \right) \\
&= -g^{\alpha\beta}(x) \Gamma_{\nu\rho}^\mu(x + \psi(x)) \frac{\partial \psi^\nu}{\partial x^\alpha} \left(\frac{\partial \psi^\rho}{\partial x^\beta} + 2\delta_\beta^\rho \right) \\
&\quad - \underbrace{(g^{\alpha\beta}(x) - g^{\alpha\beta}(x + \psi(x))) \Gamma_{\alpha\beta}^\mu(x + \psi(x))}_0 . \quad (2.7)
\end{aligned}$$

Here we have added the last, vanishing term to show that the last line can be estimated, almost everywhere, by a multiple of $|\psi|$ when the metric is Lipschitz.

We have:

LEMMA 2.8 *If $\psi^\mu = 0$ on $\mathcal{S}_{\mathcal{O};\tau}$, then $\psi^\mu = 0$ on the domain of dependence $\mathcal{D}_J(\mathcal{S}_{\mathcal{O};\tau}, \mathcal{O})$.*

PROOF: The argument proceeds via a standard energy inequality, but some care is needed to take into account the low differentiability, and the fact that (2.7) only holds in local coordinates. Let $\mathcal{S}_{\mathcal{O};\tau,n} \subset \mathcal{S}_{\mathcal{O};\tau}$ be an exhaustion of $\mathcal{S}_{\mathcal{O};\tau}$ by compact submanifolds with smooth boundary. Let X be any differentiable timelike vector field on \mathcal{M} and let T be the energy-momentum tensor associated with ψ , defined as

$$T_{\mu\nu} := \sum_{\alpha} (\partial_\mu \psi^\alpha \partial_\nu \psi^\alpha - \frac{1}{2} g^{\sigma\rho} \partial_\sigma \psi^\alpha \partial_\rho \psi^\alpha g_{\mu\nu} + \psi^\alpha \psi^\alpha X_\mu X_\nu) .$$

Then T is locally Lipschitz.

Consider the domain of dependence $\mathcal{D}_J(\mathcal{S}_{\mathcal{O};\tau,n}, \mathcal{O})$, this is a set with Lipschitz boundary. For $t \geq \tau$ set

$$\Omega_{n,t} = \mathcal{D}_J(\mathcal{S}_{\mathcal{O};\tau,n}, \mathcal{O}) \cap \{\tau \leq x^0 \leq t\} .$$

Since Ψ is the identity on $\mathcal{S}_{\mathcal{O};\tau}$, there exists $T > 0$ such that $\Psi(\Omega_{n,t}) \subset \mathcal{O}$. Hence (2.7) applies on $\Omega_{n,t}$ for $\tau \leq t \leq T$ and so there exists a constant C such that there we have, almost everywhere,

$$\begin{aligned}
|\nabla_\mu T^\mu{}_\nu| &= \left| \sum_{\alpha} \Box_g \psi^\alpha \partial_\nu \psi^\alpha + \nabla_\mu (\psi^\alpha \psi^\alpha X^\mu X_\nu) \right| \\
&\leq C(|\psi|^2 + |\partial\psi|^2) .
\end{aligned}$$

As already pointed-out, the last term in (2.7) has been estimated by $C|\psi|$ using the fact that the metric is Lipschitz-continuous; the estimation of the remaining terms is straightforward, for example the terms $(\partial\psi)^3$ are estimated by $C|\partial\psi|$. Letting

$$E_n(t) = \int_{\mathcal{S}_{\mathcal{O};t} \cap \Omega_{n,\tau}} T^\mu{}_\nu X^\nu dS_\nu ,$$

and using the Stokes' theorem for Lipschitz vector fields on Lipschitz domains [15] (see also, e.g., [13, 23]) one obtains, for some constant C_n ,

$$\forall \tau \leq t \leq T \quad E_n(t) \leq C_n \int_\tau^t E_n(s) ds .$$

Here we have used that $\psi = 0$ on $\mathcal{S}_{\mathcal{O};\tau}$ and, as before, $\partial\psi = 0$ on $\mathcal{S}_{\mathcal{O};\tau}$ as well. Gronwall's Lemma gives $E_n(t) = 0$ for $0 \leq t \leq T$. An open-closed argument shows now that $E_n(t) = 0$ for all t , hence $\psi = 0$ on $\mathcal{D}_J(\mathcal{S}_{\mathcal{O};\tau,n}, \mathcal{O})$. Since

$$\cup_n \mathcal{D}_J(\mathcal{S}_{\mathcal{O};\tau,n}, \mathcal{O}) = \mathcal{D}_J(\mathcal{S}_{\mathcal{O};\tau}, \mathcal{O}) ,$$

the result follows. \square

Returning to the proof of Proposition 2.6, let h be any complete Riemannian metric on \mathcal{M} . Let $p \in \mathcal{S}$, denote by $B_p(n)$ the open h -distance ball centred at p of radius n , and let

$$K_n := \overline{J^-(B_p(n))} \cap \overline{J^+(B_p(n))} .$$

For smooth metrics it is a standard fact that the interior of K_n is a globally hyperbolic compact subset of \mathcal{M} , with Cauchy surface $\mathcal{S} \cap K_n$; this can be seen to remain true for continuous metrics using the results in [11]. We have

$$\cup_n K_n = \mathcal{M} .$$

Let $q \in \mathcal{M}$, we want to show that $\Psi(q) = q$. There exists n such that $q \in K_n$. Since K_n is compact, it can be covered by a finite number of conditionally compact wave-coordinates patches \mathcal{U}_ℓ in which the metric is Lipschitz-continuous.

Choose any smooth differentiable structure on \mathcal{M} compatible with the C^1 atlas in which the metric is continuous. By [11] or [14] there exists a smooth Cauchy time function t on \mathcal{M} so that $\mathcal{S} = \{t = 0\}$. Set

$$I_n := \{\tau \in \mathbb{R} : \Psi = \text{id on } \mathcal{S}_\tau \cap K_n\} ,$$

where \mathcal{S}_τ denotes the τ -level set of t . (To avoid ambiguities, we consider that $t \in I_n$ when $\mathcal{S}_t \cap K_n$ is empty.) Then $I_n \neq \emptyset$ as $0 \in I_n$, and I_n is clearly closed in \mathbb{R} . We wish to show that $I_n = \mathbb{R}$, hence Ψ is the identity on K_n . For this, it remains to show that I_n is open.

Let then $t \in I_n$, and consider those \mathcal{U}_ℓ 's that intersect \mathcal{S}_t , renumbering we can assume that this happens for $\ell = 1, \dots, N$ for some $N = N(t)$. Then Ψ is the identity on $\mathcal{S}_{\mathcal{U}_\ell;t}$, and so Ψ is the identity on $\mathcal{D}_J(\mathcal{S}_{\mathcal{U}_\ell;t}, \mathcal{U}_\ell)$ by Lemma 2.8. Hence ψ is the identity on $\mathcal{D}_J(\mathcal{S}_t \cap K_n, \cup_{\ell=1}^N \mathcal{U}_\ell)$. Compactness of K_n implies that for t' close enough to t we have $\mathcal{S}_{t'} \cap K_n \subset \cup_{\ell=1}^N \mathcal{U}_\ell$, which establishes openness of I_n , and finishes the proof of Proposition 2.6. \square

One can use [18, Theorem III] to cover a manifold with a $H_{\text{space,loc}}^s$ metric, $s > n/2 + 1$, by wave-map coordinate patches. However, when transformed to wave-coordinates, the metric will be of $H_{\text{space,loc}}^{s-1}$ -differentiability class only in general. The requirement of existence of the embedding $H^{s-1} \subset C^{0,1}$ leads to the threshold $s > n/2 + 2$ for the applicability of Proposition 2.6 for general $H_{\text{space,loc}}^s$ metrics.

On the other hand, solutions of the vacuum Einstein equations can be constructed directly by patching-together domains of definition of wave-coordinates [26, 27], and then Proposition 2.6 applies without loss of differentiability for the metric when $s > n/2 + 1$.

3 Global uniqueness

3.1 An abstract theorem

In the context of $H_{\text{space,loc}}^s$ -Lorentzian manifolds, a hypersurface S will be said to be *compatible* if S is the image of a coordinate-level set of a diffeomorphism of $H_{\text{space,loc}}^{s+1}$ -differentiability class.

We will prove a somewhat more general version of Theorem 1.1, where the differentiability index $s \in \mathbb{R}$ is only assumed to satisfy $s > n/2$, as needed to ensure continuity of the metric,² provided that the following holds:

- H1. The harmonically-reduced vacuum Einstein equations with initial data in $H_{\text{loc}}^s(S) \oplus H_{\text{loc}}^{s-1}(S)$, $S \subset \{t = 0\} \subset \mathbb{R}^{n+1}$ have local solutions.
- H2. Two solutions g_1 and g_2 in $H_{\text{space,loc}}^s$ globally coordinatized by harmonic coordinates with the same data on $S \subset \{t = 0\} \subset \mathbb{R}^{n+1}$ coincide on $\mathcal{D}_{J,g_1}(S) \cap \mathcal{D}_{J,g_2}(S)$.
- A0. A time-orientation-preserving C^1 isometry of $g \in H_{\text{space,loc}}^s$ which is the identity on the initial data hypersurface is the identity everywhere.
- A1. Let Φ be a C^1 isometry of two $H_{\text{space,loc}}^s$ Lorentzian manifolds. Then Φ is of $H_{\text{space,loc}}^{s+1}$ -differentiability class.
- A2. For any compatible spacelike acausal hypersurface S and for any $H_{\text{space,loc}}^{s+1}$ function ϕ the Cauchy problem

$$\square_g f = 0, \quad f|_S = \phi|_S, \quad \partial f|_S = \partial \phi|_S,$$

has a unique solution of $H_{\text{space,loc}}^{s+1}$ -differentiability class in the J -domain of dependence of S .

- A3. If Ψ is of $H_{\text{space,loc}}^{s+1}$ -differentiability class and g is in $H_{\text{space,loc}}^s$, then Ψ^*g is in $H_{\text{space,loc}}^s$.

The logic for the numbering of the hypotheses is the following: H1-H2 are needed for local existence and uniqueness of solutions near the initial data hypersurface. Hypothesis A0 is used in the proof existence of maximal developments. Hypotheses A1-A3 are then the supplementary hypotheses needed for proving uniqueness of maximal globally hyperbolic developments.

We claim that:

THEOREM 3.1 *Under the remaining hypotheses of Theorem 1.1, suppose instead that $\mathbb{R} \ni s > n/2$. If moreover the hypotheses H1, H2, A0, A1, A2 and A3 hold, then the conclusions of Theorem 1.1 hold.*

The key to the proof is the following Proposition:

PROPOSITION 3.2 *Let $s > n/2$, let (\mathcal{M}_a, g_a) , $a = 1, 2$, be vacuum globally hyperbolic $H_{\text{space,loc}}^s$ space-times with C^1 Cauchy surfaces \mathcal{S}_a and with $H_{\text{space,loc}}^s \oplus H_{\text{space,loc}}^{s-1}$ initial data, and suppose that (\mathcal{M}_2, g_2) is maximal.*

²Minor variations of our conditions would apply to metrics which are of $H_{\text{space,loc}}^s \cap C^0$ -differentiability class, with $s \leq n/2$. Similarly one could use more general spaces of metrics and maps, with correspondingly modified conditions H1-A3.

Let $\mathcal{O} \subset \mathcal{M}_1$ be a connected neighborhood of \mathcal{S}_1 and suppose there exists a one-to-one time-orientation preserving isometry $\Psi_{\mathcal{O}} : \mathcal{O} \rightarrow \mathcal{M}_2$, such that $\Psi_{\mathcal{O}}(\mathcal{S}_1)$ is acausal. If the hypotheses H2 and A1-A3 hold, then there exists a one-to-one isometry

$$\Psi : \mathcal{M}_1 \rightarrow \mathcal{M}_2, \quad (3.1)$$

such that $\Psi|_{\mathcal{O}} = \Psi_{\mathcal{O}}$.

Before passing to the proof of Proposition 3.2, let us note that Theorem 3.1 is a corollary thereof:

PROOF OF THEOREM 3.1: The existence of some vacuum globally hyperbolic development is a standard consequence of hypotheses H1, H2, A2 and A3. The existence of maximal vacuum globally hyperbolic developments follows from hypothesis A0 and Theorem 2.2. The fact that any two maximal vacuum globally hyperbolic developments are isometrically diffeomorphic follows immediately from Proposition 3.2. \square

To complete the proof of Theorem 3.1 it remains to prove Proposition 3.2:

PROOF OF PROPOSITION 3.2: The condition that $s > n/2$ guarantees that the metric is continuous, and so the causality theory of [11] with continuous metrics applies.

Consider the collection \mathcal{X} of all pairs $(\mathcal{U}, \Psi_{\mathcal{U}})$, where $\mathcal{U} \subset \mathcal{M}_1$ is a neighborhood of \mathcal{S}_1 such that \mathcal{S}_1 is a Cauchy surface for $(\mathcal{U}, g_1|_{\mathcal{U}})$,³ and where $\Psi_{\mathcal{U}} : \mathcal{U} \rightarrow \mathcal{M}_2$ is an isometric diffeomorphism between \mathcal{U} and $\Psi_{\mathcal{U}}(\mathcal{U}) \subset \mathcal{M}_2$ satisfying

$$\Psi_{\mathcal{U}}|_{\mathcal{S}_1} = \Psi_{\mathcal{O}}|_{\mathcal{S}_1}. \quad (3.2)$$

The collection \mathcal{X} can be partially ordered by inclusion: $(\mathcal{U}, \Psi_{\mathcal{U}}) \leq (\mathcal{V}, \Psi_{\mathcal{V}})$ if $\mathcal{U} \subset \mathcal{V}$ and if $\Psi_{\mathcal{V}}|_{\mathcal{U}} = \Psi_{\mathcal{U}}$. Let $(\mathcal{U}_{\alpha}, \Psi_{\alpha})_{\alpha \in \Omega}$ be a chain in \mathcal{X} , set $\mathcal{W} = \cup_{\alpha \in \Omega} \mathcal{U}_{\alpha}$, define $\Psi_{\mathcal{W}} : \mathcal{W} \rightarrow \mathcal{M}_2$ by $\Psi_{\mathcal{W}}|_{\mathcal{U}_{\alpha}} = \Psi_{\alpha}$; clearly $(\mathcal{W}, \Psi_{\mathcal{W}})$ is a majorant for $(\mathcal{U}_{\alpha}, \Psi_{\alpha})_{\alpha \in \Omega}$. As in the proof of Theorem 2.2, using the set-theory axioms from [19, Appendix] it can be seen that \mathcal{X} forms a set, we can thus apply the Kuratowski-Zorn Lemma [19] to conclude that there exist maximal elements in \mathcal{X} . Let then $(\tilde{\mathcal{M}}, \Psi)$ be any maximal element, by definition $(\tilde{\mathcal{M}}, g_1|_{\tilde{\mathcal{M}}})$ is thus globally hyperbolic with Cauchy surface \mathcal{S}_1 , and Ψ is a one-to-one isometry from $\tilde{\mathcal{M}}$ into \mathcal{M}_2 such that $\Psi|_{\mathcal{S}_1} = \Psi_{\mathcal{O}}|_{\mathcal{S}_1}$. Viewing \mathcal{O} as a subset of $\tilde{\mathcal{M}}$, Proposition 2.4 applied to $\Psi \circ \Psi_{\mathcal{O}}^{-1}$ gives

$$\Psi|_{\mathcal{O}} = \Psi_{\mathcal{O}}. \quad (3.3)$$

As a next step, we prove:

LEMMA 3.3 *Under the hypotheses of Proposition 3.2, suppose that $(\mathcal{O}, \Psi_{\mathcal{O}})$ is maximal. Then the manifold*

$$\mathcal{M}' = (\mathcal{M}_1 \sqcup \mathcal{M}_2) / \Psi_{\mathcal{O}}$$

is Hausdorff.

³By this we mean that every future-directed future-inextendible causal curve which starts in $\mathcal{U} \cap J^-(\mathcal{S}_1)$ remains in \mathcal{U} until it meets \mathcal{S}_1 ; similarly for past-directed causal curves starting in $\mathcal{U} \cap J^+(\mathcal{S}_1)$.

REMARK 3.4 Recall that \sqcup denotes the disjoint union, while $(\mathcal{M}_1 \sqcup \mathcal{M}_2)/\Psi$ is the quotient manifold $(\mathcal{M}_1 \sqcup \mathcal{M}_2)/\sim$, where $p_1 \in \mathcal{M}_1$ is equivalent to $p_2 \in \mathcal{M}_2$ if $p_2 = \Psi(p_1)$.

PROOF: Let $p', q' \in \mathcal{M}'$ be such that there exist no open neighborhoods separating p' and q' ; clearly this is possible only if, interchanging p with q if necessary, we have $p' = [p]$, with $p \in \partial\mathcal{O}$ and $q' = [q]$, with $q \in \partial\Psi_{\mathcal{O}}(\mathcal{O})$. Such points p, p', q and q' will be called “non-Hausdorff”.

Let \mathcal{H} denote the set of non-Hausdorff points p in \mathcal{M}_1 , thus $p' = i_{\mathcal{M}_1}(p) \equiv [p]$ is non-Hausdorff in \mathcal{M}' , where $i_{\mathcal{M}_1}$ denotes the embedding of \mathcal{M}_1 into \mathcal{M}' . By elementary topology \mathcal{H} is closed (as its complement is open), and we have just seen that $\mathcal{H} \subset \partial\mathcal{O}$.

Suppose that $\mathcal{H} \neq \emptyset$, changing time orientation if necessary we may assume that $\mathcal{H} \cap I^+(\mathcal{S}_1) \neq \emptyset$. Let $\hat{p} \in \mathcal{H} \cap I^+(\mathcal{S}_1)$. We wish to show that there necessarily exists $p \in \mathcal{H}$ such that

$$J^-(p) \cap \mathcal{H} \cap I^+(\mathcal{S}_1) = \{p\}. \quad (3.4)$$

If (3.4) holds with $p = \hat{p}$ we are done, otherwise consider the (non-empty) set \mathcal{Y} of future directed causal paths $\Gamma : [0, 1] \rightarrow I^+(\mathcal{S})$ such that $\Gamma(0) \in \mathcal{H}, \Gamma(1) = \hat{p}$. \mathcal{Y} is directed by inclusion: $\Gamma_1 < \Gamma_2$ if $\Gamma_1([0, 1]) \subset \Gamma_2([0, 1])$. Let $\{\Gamma_\alpha\}_{\alpha \in \Omega}$ be a chain in \mathcal{Y} , set $\Gamma = \cup_{\alpha \in \Omega} \Gamma_\alpha([0, 1])$, consider the sequence $p_\alpha = \Gamma_\alpha(0)$. Clearly $\Gamma \subset J^+(\mathcal{S}_1) = I^+(\mathcal{S}_1) \cup \mathcal{S}_1$, and global hyperbolicity implies that Γ must be extendible, thus $\Gamma_\alpha(0)$ accumulates at some $p_* \in I^+(\mathcal{S}_1) \cup \mathcal{S}_1$. As \mathcal{O} is an open neighborhood of \mathcal{S}_1 the case $p_* \in \mathcal{S}_1$ is not possible, hence $p_* \in I^+(\mathcal{S}_1)$ and consequently $\Gamma \in \mathcal{Y}$. It follows that every chain in \mathcal{Y} has a majorant, and by Zorn’s Lemma \mathcal{Y} has maximal elements. Let then Γ be any maximal element of \mathcal{Y} , setting $p = \Gamma(0)$ the equality (3.4) must hold.

We now claim that (3.4) also implies

$$J^-(p) \cap \partial\mathcal{O} \cap I^+(\mathcal{S}_1) = \{p\}. \quad (3.5)$$

In order to establish (3.5), we start with the following lemma (see [11] for terminology and notation):

LEMMA 3.5 *Let $p \in \partial\mathcal{O} \cap J^+(\mathcal{S}_1)$, then $\check{I}^-(p) \cap J^+(\mathcal{S}_1) \subset \mathcal{O}$.*

REMARK 3.6 For $C^{0,1}$ metrics one has $I^-(p) = \check{I}^-(p)$ [11], and then the result is standard. We do not know whether the inclusion $I^-(p) \cap J^+(\mathcal{S}_1) \subset \mathcal{O}$ holds for metrics which are merely continuous.

PROOF: Let $q \in \check{I}^-(p) \cap J^+(\mathcal{S}_1)$, then $p \in \check{I}^+(q)$, and so $\check{I}^+(q)$ forms an open neighborhood of p . Let $p_i \in \mathcal{O}$ be a sequence converging to p , then $p_i \in \check{I}^+(q)$ for i large enough. Let γ_i be a timelike curve from p_i to q , note that γ_i does not meet \mathcal{S}_1 since $q \in I^+(\mathcal{S}_1)$ and \mathcal{S}_1 is achronal. Let $\hat{\gamma}_i$ be any past inextendible causal extension of γ_i . Global hyperbolicity implies that $\hat{\gamma}_i$ is included in \mathcal{O} at least until it meets \mathcal{S}_1 when followed to the past from p_i , hence $q \in \mathcal{O}$. \square

Returning to the proof of (3.5), suppose that the claim is wrong, then there exists a point $q \in J^-(p) \cap \partial\mathcal{O} \cap I^+(\mathcal{S}_1)$ which is distinct from p .

Let γ_p be a past-inextendible l.u.t. curve starting at p ; by global hyperbolicity γ_p meets \mathcal{S}_1 . Similarly let γ_q be a past inextendible g -causal

curve starting at q . By Lemma 3.5 points on γ_p distinct from p , and on γ_q distinct from q , and lying to the future of \mathcal{S}_1 are in \mathcal{O} .

Let $q_i \neq q$ be any sequence of points on γ_q converging to q such that $q_{i+1} \ni \tilde{I}_{g_1}^+(q_i)$. In particular $q_i \in \mathcal{O}$. The aim of the argument is to show that $\Psi_{\mathcal{O}}(q_i)$ has a limit in \mathcal{M}_2 , which will imply that $q \in \mathcal{H}$, a contradiction with (3.4). The standard proof [9, 11] of existence of the limit of the sequence $\Psi_{\mathcal{O}}(q_i)$ for $C^{0,1}$ metrics uses Lemma 3.5 together with the fact that a causal curve which is not everywhere null can be deformed, with end points fixed, to a timelike one. However, there exist continuous Lorentzian metrics for which this is wrong [11], and a different line of thought is needed.

By [Theorem 2.8][11] there exists a smooth metric $\hat{g}_1 \succ g_1$ on \mathcal{M}_1 so that \mathcal{M}_1 is globally hyperbolic with Cauchy surface \mathcal{S}_1 . For $i \geq 2$ let \hat{g}_i be any sequence of smooth metrics converging locally uniformly to g_1 such that

$$\hat{g}_1 \succ \hat{g}_i \succ \hat{g}_{i+1} \succ g_1 .$$

Then all the spacetimes $(\mathcal{M}_1, \hat{g}_i)$ are globally hyperbolic with Cauchy surface \mathcal{S}_1 .

For any j , the closed null Lipschitz hypersurfaces $\partial J_{\hat{g}_i}^+(q_j)$ separate \mathcal{M}_1 , with p lying to their causal future and \mathcal{S}_1 lying to their past. The curve γ_p intersects each of the $\partial J_{\hat{g}_i}^+(q_j)$'s: indeed, γ_p has to exit the compact set $J_{\hat{g}_i}^-(p) \cap J_{\hat{g}_i}^+(q_j)$; since $\partial J_{\hat{g}_i}^-(p)$ is achronal, it can only do so through $\partial J_{\hat{g}_i}^+(q_j)$. One can then construct a \hat{g}_i -causal curve $\gamma_{j,i}$ from p to q_j by following γ_p from p to its intersection point with $\partial J_{\hat{g}_i}^+(q_j)$, and then following a generator of $\partial J_{\hat{g}_i}^+(q_j)$ until q_j is reached. For each j the curves $\gamma_{j,i}$ are \hat{g}_i -causal, and $(\mathcal{M}_1, \hat{g}_1)$ is globally hyperbolic, therefore there exists a \hat{g}_1 -causal curve γ_j from q_j to p which is an accumulation curve of the $\gamma_{j,i}$'s. The curve γ_j is g_1 -causal by [Theorem 1.6][11].

By Lemma 3.5 the curves γ_j are included in \mathcal{O} except for their end-point p . It is convenient to parameterize the γ_j 's by distance from q_j with respect to an auxiliary complete Riemannian metric on \mathcal{M}_2 . Let s_i be defined as $p = \gamma_i(s_i)$.

Denote by $q \in \mathcal{M}_2$ the non-Hausdorff partner of p . Then the curve in \mathcal{M}_2 defined as

$$\hat{\gamma}_i := \Psi_{\mathcal{O}} \circ \gamma_i|_{[0, s_i]}$$

is a g_2 -causal curve lying in the compact set (see [Theorem 2.9.9][9])

$$J_{g_2}^-(q) \cap J_{g_2}^+(\Psi_{\mathcal{O}}(q_1)) \subset \mathcal{M}_2 .$$

Hence $\hat{\gamma}_i$ has an accumulation point, say \hat{r}_i , lying on the boundary of $\Psi_{\mathcal{O}}(\mathcal{O})$. The points r_i and \hat{r}_i form a non-Hausdorff pair, which is only compatible with (3.4) if $\hat{r}_i = q$. So, in fact, $\hat{\gamma}_i$ can be extended to a causal curve from $\Psi_{\mathcal{O}}(q_i)$ to q by adding the end point. We will denote by the same symbol that extension.

By global hyperbolicity of \mathcal{M}_2 , passing to a subsequence if necessary, the sequence $\hat{\gamma}_i$ accumulates at a g_2 -causal curve $\hat{\gamma}$. This shows that the sequence

$$\Psi_{\mathcal{O}}(q_i) = \hat{\gamma}_i(0)$$

has a limit point in \mathcal{M}_2 . Hence $q \in \mathcal{H}$, which contradicts (3.4). We conclude that (3.5) must hold.

To continue, let $p_1 \in \mathcal{M}_1$, $p_2 \in \mathcal{M}_2$, be any non-Hausdorff pair in \mathcal{M}' such that (3.4) holds with $p = p_1$. Around p_2 we can construct harmonic coordinates y^μ as follows: Let z^μ be local coordinates defined in some neighborhood \mathcal{O}_2 of p_2 , such that the metric coefficients are of $H^s_{\text{space,loc}}$ differentiability class; such coordinates will be said to be $H^s_{\text{space,loc}}$ -compatible. We can, and will, further assume that $z^0(p_2) = 0$, and that the level sets of z^0 are spacelike and acausal near p_2 . Set

$$\mathcal{I}_\tau = \{q \in \mathcal{O}_2 : z^0(q) = \tau\}.$$

Passing to a subset of \mathcal{O}_2 if necessary we may assume that \mathcal{O}_2 is globally hyperbolic with Cauchy surface \mathcal{I}_0 . By hypothesis A1 there exist functions $y^\mu \in H^{s+1}_{\text{space,loc}}$, (unique) solutions of the problem

$$\begin{aligned} \square_{g_2} y^\mu &= 0, \\ y^0|_{\mathcal{I}_0} &= 0, \quad \frac{\partial y^0}{\partial z^0}\bigg|_{\mathcal{I}_0} = 1, \quad y^i|_{\mathcal{I}_0} = z^i, \quad \frac{\partial y^i}{\partial z^0}\bigg|_{\mathcal{I}_0} = 0, \end{aligned} \quad (3.6)$$

where \square_γ is the d'Alembert operator of a metric γ . Passing once more to a globally hyperbolic subset of \mathcal{O}_2 if necessary, the functions y^μ form a coordinate system on \mathcal{O}_2 .

Let w^μ be any $H^s_{\text{space,loc}}$ -compatible coordinates near p_1 with domain of definition \mathcal{V} . We can choose $\epsilon > 0$ such that (see Figure 3.1)

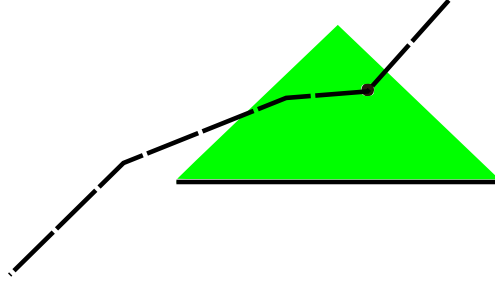


Figure 3.1: Extending the isometry $\Psi_{\mathcal{O}}$ near a spacelike point of $\partial\mathcal{O}$. The point p_2 is located at the dot, the dashed line is $\partial\mathcal{O}$, the set \mathcal{O} lies under that line. The shaded region is the future domain of dependence of $\hat{\mathcal{I}}$.

1. $\mathcal{D}_J^+(\mathcal{I}_{-\epsilon}) \subset \mathcal{O}_2$,
2. $p_2 \in \text{int}\mathcal{D}_J^+(\mathcal{I}_{-\epsilon})$,
3. $\overline{\mathcal{I}_{-\epsilon}} \subset \Psi_{\mathcal{O}}(\mathcal{O})$,
4. $\hat{\mathcal{I}} := \Psi_{\mathcal{O}}^{-1}(\mathcal{I}_{-\epsilon}) \subset \mathcal{V}$.

Now, $\Psi_{\mathcal{O}}^{-1}$ is of $H^{s+1}_{\text{space,loc}}$ -differentiability class by hypothesis A1 and $\hat{\mathcal{I}}$ is the image of the $(-\epsilon)$ -level set of the coordinate z^0 by $\Psi_{\mathcal{O}}^{-1}$. We can thus invoke hypothesis A2 to define on $\mathcal{D}_J(\hat{\mathcal{I}})$ the functions $x^\mu \in H^{s+1}_{\text{space,loc}}(\mathcal{D}_J(\hat{\mathcal{I}}))$ as the unique solutions of the problem

$$\begin{aligned} \square_{g_1} x^\mu &= 0, \\ x^\mu|_{\hat{\mathcal{I}}} &= y^\mu \circ \Psi_{\mathcal{O}}|_{\hat{\mathcal{I}}}, \quad \frac{\partial x^\mu}{\partial \hat{n}}\bigg|_{\hat{\mathcal{I}}} = \frac{\partial (y^\mu \circ \Psi_{\mathcal{O}})}{\partial \hat{n}}\bigg|_{\hat{\mathcal{I}}}, \end{aligned}$$

where $\frac{\partial}{\partial \hat{n}}$ is the derivative in the direction normal to $\hat{\mathcal{I}}$.

By isometry-invariance and by the uniqueness part of hypothesis A2 we have

$$x^\mu|_{\mathcal{D}_J(\hat{\mathcal{I}}) \cap \mathcal{O}} = y^\mu \circ \Psi|_{\mathcal{O}|_{\mathcal{D}_J(\hat{\mathcal{I}}) \cap \mathcal{O}}} . \quad (3.7)$$

Equivalently, when expressed in terms of local coordinates x^μ near p_1 and y^μ near p_2 , the map $\Psi_{\mathcal{O}}$ is the identity on $\mathcal{D}_J(\hat{\mathcal{I}}) \cap \mathcal{O}$. In particular the x^μ 's form a coordinate system on $\mathcal{D}_J(\hat{\mathcal{I}}) \cap \mathcal{O}$. Since $\Psi_{\mathcal{O}}$ is an isometry by hypothesis, on $\mathcal{D}_J(\hat{\mathcal{I}}) \cap \mathcal{O}$ the metric functions for the metric g_1 , when expressed in the coordinates x^μ coincide with the metric functions for the metric g_1 , when expressed in the coordinates y^μ .

On $\mathcal{O} \cap \mathcal{V}$ we have, by (3.7),

$$\begin{aligned} g_1(w)_{\mu\nu} dw^\mu dw^\nu &= g_2(y(w))_{\alpha\beta} \frac{\partial y^\alpha}{\partial w^\mu} \frac{\partial y^\beta}{\partial w^\nu} dw^\mu dw^\nu \\ &= g_2(y(w))_{\alpha\beta} \frac{\partial x^\alpha}{\partial w^\mu} \frac{\partial x^\beta}{\partial w^\nu} dw^\mu dw^\nu , \end{aligned}$$

hence

$$\det \left(\frac{\partial x^\alpha}{\partial w^\mu} \right)^2 \Big|_{\mathcal{O} \cap \mathcal{V}} = \frac{\det (g_1(w)_{\mu\nu})}{\det (g_2(y(w))_{\alpha\beta})} \Big|_{\mathcal{O} \cap \mathcal{V}} .$$

Since the right-hand side is uniformly bounded away from zero on \mathcal{O} , continuity shows that $\frac{\partial x^\alpha}{\partial w^\mu}$ does not vanish at p_1 . By the implicit function theorem there exists a neighborhood

$$\mathcal{W} \subset \mathcal{D}_J^+(\hat{\mathcal{I}})$$

of p_1 such that the map $\mathcal{W} \ni w^\mu \mapsto x^\mu$ is a diffeomorphism onto its image.

Let

$$\tilde{\mathcal{I}}_t := \{x^0 = t\} \subset \mathcal{W}$$

Making \mathcal{W} smaller if necessary, we can choose $\eta > 0$ small enough so that $\tilde{\mathcal{I}}_{-\eta}$ satisfies

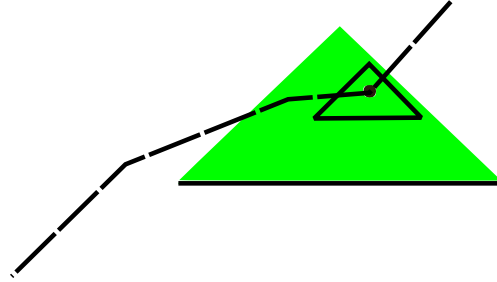


Figure 3.2: The functions x^μ form a wave-coordinate system on the future domain of dependence of $\tilde{\mathcal{I}}_{-\eta}$, which is represented by the triangle around p_2 .

1. $p_1 \in \text{int} \mathcal{D}_J^+(\tilde{\mathcal{I}}_{-\eta})$,
2. $\overline{\tilde{\mathcal{I}}_{-\eta}} \subset \mathcal{O}$.

Set

$$\mathcal{U} = \mathcal{O} \cup \mathcal{D}_J^+(\tilde{\mathcal{I}}_{-\eta}) ,$$

and for $p \in \mathcal{U}$ define

$$\Psi_{\mathcal{U}}(p) = \begin{cases} \Psi_{\mathcal{O}}(p), & p \in \mathcal{O}, \\ q : \text{where } q \text{ is such that } x^\mu(q) = y^\mu(p), & p \in \mathcal{D}_J^+(\tilde{\mathcal{I}}_{-\eta}) \setminus \mathcal{O}. \end{cases} \quad (3.8)$$

From what has been said, defines a $H_{\text{space,loc}}^{s+1}$ map from \mathcal{U} to \mathcal{M}_2 .

Clearly \mathcal{U} is a globally hyperbolic neighborhood of \mathcal{S}_1 , and \mathcal{S}_1 is a Cauchy surface for \mathcal{U} . Note that \mathcal{O} is a proper subset of \mathcal{U} , as $p_1 \in \mathcal{D}_J^+(\tilde{\mathcal{I}}_{-\eta})$ but $p_1 \notin \mathcal{O}$.

By construction the metric $\Psi_{\mathcal{U}}^* g_2$ is of $H_{\text{space,loc}}^s$ differentiability class. (This holds by hypothesis on \mathcal{O} , since there $\Psi_{\mathcal{U}}^* g_2$ coincides with g_1 . This holds away from \mathcal{O} as well, since there the map Ψ is the identity in local coordinates, and in those the metric has already been shown to be in $H_{\text{space,loc}}^s$). The metric $\Psi_{\mathcal{U}}^* g_2$ coincides with g_1 near $\tilde{\mathcal{I}}_{-\eta}$. It follows from hypothesis $H2$ that $\Psi_{\mathcal{U}}^* g_2$ coincides with g_1 on $\mathcal{U} \cap \mathcal{D}_J^+(\tilde{\mathcal{I}}_{-\eta})$. So $\Psi_{\mathcal{U}}$ is a local isometry.

To prove that $\Psi_{\mathcal{U}}$ is one-to-one, we proceed by contradiction, and consider $p, q \in \mathcal{U}$, $p \neq q$, such that

$$\Psi_{\mathcal{U}}(p) = \Psi_{\mathcal{U}}(q). \quad (3.9)$$

Since $\Psi_{\mathcal{O}}$ is one-to-one, and since the map

$$x^\mu \mapsto y^\mu \quad (3.10)$$

constructed above in local coordinates is one-to-one, (3.9) can only occur with $p \neq q$ if p lies in the domain of the map (3.10) and q lies in \mathcal{O} , or vice-versa. Exchanging p and q if necessary, we only need to consider the former case, and note that $p \notin \mathcal{O}$ since then $\Psi_{\mathcal{U}}$ would coincide with $\Psi_{\mathcal{O}}$ near p , and would therefore be injective there. So p must lie in the complement of \mathcal{O} , but $\Psi_{\mathcal{U}}(p)$ must lie in $\Psi_{\mathcal{O}}(\mathcal{O})$.

Consider a past directed timelike curve Γ_1 entirely contained in \mathcal{O} , inextendible in \mathcal{O} , and passing through q . Set $\Gamma := \Psi_{\mathcal{O}}(\Gamma_1)$. Since the map (3.10) is a local diffeomorphism, we can invert it locally to obtain a pre-image of Γ which is a past-directed timelike curve Γ_2 through p . Suppose that Γ_2 meets $\mathcal{I}_{-\epsilon} \subset \mathcal{O}$ when followed to the past. Since $\Psi_{\mathcal{O}}$ is one-to-one, the part of Γ_2 that lies in \mathcal{O} must coincide with Γ_1 , which is not possible since Γ_1 has an end-point at q , while Γ_2 leaves \mathcal{O} through $\partial\mathcal{O}$. We infer that Γ_2 stops before meeting $\mathcal{I}_{-\epsilon} \subset \mathcal{O}$ when followed to the past. So global hyperbolicity implies that Γ must meet \mathcal{S}_1 when followed to the future. One can then construct a timelike curve from $\Psi_{\mathcal{O}}(\mathcal{S}_1)$ to itself by following the image by $\Psi_{\mathcal{U}}$ of any causal curve from \mathcal{S}_1 to $\Psi_{\mathcal{O}}(p)$, and then Γ from $\Psi_{\mathcal{O}}(p)$ to $\Psi_{\mathcal{U}}$, which is not possible as we assumed that $\Psi_{\mathcal{U}}$ is achronal. This shows that no distinct points p and q satisfying (3.9) exist, and we conclude that $\Psi_{\mathcal{O}}$ is injective, as desired.

We have thus shown, that $(\mathcal{O}, \Psi_{\mathcal{O}}) \leq (\mathcal{U}, \Psi_{\mathcal{U}})$ and $(\mathcal{O}, \Psi_{\mathcal{O}}) \neq (\mathcal{U}, \Psi_{\mathcal{U}})$ which contradicts maximality of $(\mathcal{O}, \Psi_{\mathcal{O}})$. It follows that \mathcal{M}' is Hausdorff, establishing Lemma 3.3. \square

Returning to the proof of Proposition 3.2, let $(\tilde{\mathcal{M}}, \Psi)$ be maximal. If $\tilde{\mathcal{M}} = \mathcal{M}_1$ we are done, suppose then that $\tilde{\mathcal{M}} \neq \mathcal{M}_1$. Consider the manifold

$$\mathcal{M}' = (\mathcal{M}_1 \sqcup \mathcal{M}_2) / \Psi.$$

By Lemma 3.3, \mathcal{M}' is Hausdorff. We claim that \mathcal{M}' is globally hyperbolic with Cauchy surface $\mathcal{S}' = i_{\mathcal{M}_2}(\mathcal{S}_2) \approx \mathcal{S}_2$ (recall that $i_{\mathcal{M}_a}$ denotes the canonical embedding of \mathcal{M}_a in \mathcal{M}'). Indeed, let $\Gamma' \subset \mathcal{M}'$ be an inextendible causal curve in \mathcal{M}' , set $\Gamma_1 = i_{\mathcal{M}_1}^{-1}(\Gamma' \cap i_{\mathcal{M}_1}(\mathcal{M}_1))$, $\Gamma_2 = i_{\mathcal{M}_2}^{-1}(\Gamma' \cap i_{\mathcal{M}_2}(\mathcal{M}_2))$. Clearly $\Gamma_1 \cup \Gamma_2 \neq \emptyset$, so that either $\Gamma_1 \neq \emptyset$, or $\Gamma_2 \neq \emptyset$, or both. Let the index a be such that $\Gamma_a \neq \emptyset$. If $\hat{\Gamma}_a$ were an extension of Γ_a in \mathcal{M}_a , then $i_{\mathcal{M}_a}(\hat{\Gamma}_a)$ would be an extension of Γ' in \mathcal{M}' , which contradicts maximality of Γ' , thus Γ_a is inextendible. Suppose that $\Gamma_1 \neq \emptyset$; as Γ_1 is inextendible in \mathcal{M}_1 we must have $\Gamma_1 \cap \mathcal{S}_1 = \{p_1\}$ for some $p_1 \in \mathcal{S}_1$. We then have $\Psi(p_1) \in \Gamma_2$, so that it always holds that $\Gamma_2 \neq \emptyset$. By global hyperbolicity of \mathcal{M}_2 and inextendibility of Γ_2 it follows that $\Gamma_2 \cap \mathcal{S}_2 = \{p_2\}$ for some $p_2 \in \mathcal{S}_2$, hence $\Gamma' \cap i_{\mathcal{M}_2}(\mathcal{S}_2) = \{i_{\mathcal{M}_2}(p_2)\}$. This shows that $i_{\mathcal{M}_2}(\mathcal{S}_2)$ is a Cauchy surface for \mathcal{M}' , thus \mathcal{M}' is globally hyperbolic. As $\mathcal{M} \neq \mathcal{M}_1$ we have $\mathcal{M}' \neq \mathcal{M}_2$ which contradicts maximality of \mathcal{M}_2 . It follows that we must have $\mathcal{M} = \mathcal{M}_1$, and Proposition 3.2 follows. \square

3.2 Proof of Theorem 1.1

We are ready now to pass to the proof of Theorem 1.1; this occupies the remainder of this section. In view of Theorem 3.1, we need to check that conditions H1–A3, p. 10, are satisfied when $\mathbb{N} \ni s > n/2 + 1$.

The hypotheses H1 holds by [18], while H2 can be established using energy arguments along the lines of the proof of Lemma 2.8.

The hypothesis A0 follows from the embedding $H_{\text{space,loc}}^s \subset C^{0,1}$ for $s > n/2 + 1$.

Condition A1 is the contents of the following Proposition:

PROPOSITION 3.7 *Consider a local diffeomorphism Ψ of C^1 -differentiability class, and a metric g of $H_{\text{space,loc}}^s$ -differentiability class in a coordinate system y^μ , with $\mathbb{N} \ni s > n/2 + 1$. Let*

$$\bar{g} := \Psi^* g ,$$

If \bar{g} is also of $H_{\text{space,loc}}^s$ -differentiability class with respect to a coordinate system x^μ , then

$$\Psi \in H_{\text{space,loc}}^{s+1} . \quad (3.11)$$

PROOF: In local coordinates so that $y^\mu = \Psi^\mu(x^\alpha)$, (2.3)-(2.4) take the form

$$\bar{g}_{\alpha\beta}(x) = g_{\mu\nu}(\Psi(x)) \frac{\partial \Psi^\mu}{\partial x^\alpha} \frac{\partial \Psi^\nu}{\partial x^\beta} , \quad (3.12)$$

$$\frac{\partial^2 \Psi^\mu}{\partial x^\alpha \partial x^\beta} = \bar{\Gamma}_{\alpha\beta}^\sigma(x) \frac{\partial \Psi^\mu}{\partial x^\sigma} - \Gamma_{\nu\rho}^\mu(\Psi(x)) \frac{\partial \Psi^\nu}{\partial x^\alpha} \frac{\partial \Psi^\rho}{\partial x^\beta} , \quad (3.13)$$

where the $\bar{\Gamma}_{\alpha\beta}^\sigma$'s are the Christoffel symbols of \bar{g} . Since both g and \bar{g} are in $C^{0,1}$, as in the proof of Proposition 2.6 we have $\Psi \in C^{1,1} = W^{2,\infty}$.

For $u < s - 1 - n/2$ we have $H_{\text{loc}}^{s-1} \subset C^u$ in dimension n , and a straightforward bootstrap of (3.13) shows that

$$\Psi \in C^{j_m} \subset W_{\text{loc}}^{j_m, \infty} , \quad \text{where } j_m \text{ is the largest integer strictly less than } s + 1 - \frac{n}{2} .$$

Let u_m be the largest integer strictly less than $s - 1 - \frac{n}{2}$, thus $\Psi \in C^{u_m+2}$.

Suppose, first, that $u_m + 1 \neq s - 1 - \frac{n}{2}$. For $1 \leq j \leq s - 1 - u_m$ we then have

$$H_{\text{loc}}^{s-1} \subset \cap_{1 \leq j \leq s-1-u_m} W_{\text{loc}}^{u_m+j, v_j}, \text{ where } v_j = \frac{2n}{n-2(s-1-u_m-j)} > 2. \quad (3.14)$$

Suppose that

$$\Psi \in W_{\text{space,loc}}^{u_m+j, v_{j-1}} \text{ for some } j \in \mathbb{N} \text{ satisfying } 2 \leq j \leq s - u_m. \quad (3.15)$$

Since $C^{u_m+2} \subset W_{\text{space,loc}}^{u_m+j, v_{j-1}}$, we have shown that (3.15) is true for $j = 2$. One verifies that for $s > 1 + n/2$ Lemma A.1 applies with $(\ell, q) = (u_m + j, v_j)$ and $(k + 1, p) = (u_m + j, v_{j-1})$, establishing that the map $x \mapsto \Gamma_{\nu\rho}^\mu(\Psi(x))$ is in $W_{\text{space,loc}}^{u_m+j, v_j}$. We can thus apply Lemma A.2 with $p = v_{j-1}$, $q = v_j$, $k = \ell = u_m + j$, and $m = 1$ to conclude that the map $x \mapsto \Gamma_{\nu\rho}^\mu(\Psi(x)) \frac{\partial \Psi^\nu}{\partial x^\alpha}$ is in $W_{\text{space,loc}}^{u_m+j-1, v_j}$. One similarly finds that the maps $x \mapsto \Gamma_{\nu\rho}^\mu(\Psi(x)) \frac{\partial \Psi^\nu}{\partial x^\alpha} \frac{\partial \Psi^\rho}{\partial x^\beta}$ and $x \mapsto \bar{\Gamma}_{\alpha\beta}^\sigma(x) \frac{\partial \Psi^\mu}{\partial x^\sigma}$ are in $W_{\text{space,loc}}^{u_m+j-1, v_j}$. It follows from (3.13) that $\Psi \in W_{\text{space,loc}}^{u_m+j+1, v_j}$. In a finite number of steps one obtains (3.11). In particular the proof is complete for odd space-dimensions n .

For even n we necessarily have $s \geq 2 + n/2$, and it remains to consider the case $u_m + 1 = s - 1 - \frac{n}{2}$. For $1 \leq j \leq s - 1 - u_m$ we then have

$$H_{\text{loc}}^{s-1} \subset \begin{cases} \cap_{p \in [1, \infty)} W_{\text{loc}}^{u_m+1, p}, & u_m = 0; \\ \cap_{2 \leq j \leq s-1-u_m} W_{\text{loc}}^{u_m+j, v_j} \cap_{p \in [1, \infty)} W_{\text{loc}}^{u_m+1, p}, & u_m \geq 1, \end{cases}$$

with v_j as in (3.14). Recall that we already know that $\Psi \in C^{u_m+2} \subset W_{\text{space,loc}}^{u_m+2, p}$ for any $p \in \mathbb{R}$. We can thus choose p large enough so that Lemma A.1 with $(\ell, q) = (u_m + 2, v_2)$, and $(k + 1, p) = (u_m + 2, p)$ applies, establishing that the map $x \mapsto \Gamma_{\nu\rho}^\mu(\Psi(x))$ is in $W_{\text{space,loc}}^{u_m+j, v_j}$ with $j = 2$. By Lemma A.2 with $p = q = v_j$, $k = \ell = u_m + j$, $j = 2$ and $m = 1$ the map $x \mapsto \Gamma_{\nu\rho}^\mu(\Psi(x)) \frac{\partial \Psi^\nu}{\partial x^\alpha}$ is in $W_{\text{space,loc}}^{u_m+j-1, v_j}$. One similarly finds that the maps $x \mapsto \bar{\Gamma}_{\alpha\beta}^\sigma(x) \frac{\partial \Psi^\mu}{\partial x^\sigma}$ and $x \mapsto \Gamma_{\nu\rho}^\mu(\Psi(x)) \frac{\partial \Psi^\nu}{\partial x^\alpha} \frac{\partial \Psi^\rho}{\partial x^\beta}$ are in $W_{\text{space,loc}}^{u_m+j-1, v_j}$ with $j = 2$. It follows from (3.13) that $\Psi \in W_{\text{space,loc}}^{u_m+j+1, v_j} \subset W_{\text{space,loc}}^{u_m+j+1, v_{j+1}}$ with $j = 2$. One can now continue the previous induction argument to obtain (3.11). \square

To verify A2, near S we transform the metric to a $H_{\text{space,loc}}^{s+1}$ coordinate system where S is given by the equation $\{x^0 = 0\}$. The metric is of $H_{\text{space,loc}}^s$ differentiability class in this coordinate system by A3, which has already been established. We can then obtain a local solution in local coordinates by [18]. In the overlap the solutions coincide by an energy argument, as in the proof of Lemma 2.8. The globalization to the whole domain of dependence, within a single coordinate chart, is then standard.

The hypothesis A3 follows from point 2 of Proposition A.3.

This completes the proof of Theorem 1.1. \square

A Manifolds of $W_{\text{space,loc}}^{k+1, p}$ differentiability class

When using wave-coordinates for a $H_{\text{space,loc}}^s$ metric one needs to work with coordinate transformations which are not smooth but of $H_{\text{space,loc}}^{s+1}$

differentiability class. This begs the question, what happens with functions and tensor under such coordinate changes. This is the main issue addressed in this appendix.

For $s \in \mathbb{R}$, $s > n/2 + 1$, a $W_{\text{space,loc}}^{s+1,p}$ manifold is defined as a differentiable manifold equipped with an atlas with transition functions which are in $W_{\text{space,loc}}^{s+1,p}$.

Consider a smooth manifold \mathcal{M} ; on such a manifold one can define in an invariant way tensor fields which are of C^∞ differentiability class, or of C^k class, or of $W_{\text{space,loc}}^{k,p}$ class. For example, one says that a tensor field is of $W_{\text{space,loc}}^{k,p}$ class if there exists a covering of \mathcal{M} by coordinate patches such that the coordinate components of the tensor in question are in $W_{\text{space,loc}}^{k,p}$ in each of the coordinate patches. Since the transition functions when going from one coordinate system to another are smooth, this property will be true in any coordinate system.

It is convenient to introduce the following notation: let $x, y \in \bar{\mathbb{R}}$, let us write $x \succ_* y$ if the following holds:

$$x \succ_* y \iff \begin{cases} x \geq y, & \text{if } y > 0, \\ x > y, & \text{if } y \leq 0. \end{cases} \quad (\text{A.1})$$

(Note that for $x \geq 0$ the only value of x at which “ \succ_* ” does not coincide with “ \geq ” is $x = 0$.) In this notation the Sobolev embedding theorem, in dimension n , can be stated as [1, 3]:

$$W_{\text{space,loc}}^{s,t} \subset W_{\text{space,loc}}^{u,v} \iff u \leq s \text{ and } \frac{1}{v} \succ_* \frac{1}{t} - \frac{s-u}{n}. \quad (\text{A.2})$$

We have the following:

LEMMA A.1 *Let Ω, \mathcal{U} be open subsets of \mathbb{R}^{n+1} and let $\Psi : \Omega \rightarrow \mathcal{U}$ be a C^1 diffeomorphism such that $\psi \in W_{\text{space,loc}}^{k+1,p}(\Omega; \bar{\mathbb{R}}^n)$, $p \in [1, \infty]$, $k \in \mathbb{N}$, $kp > n$. If (ℓ, q) is such that the n -dimensional Sobolev embedding $W_{\text{space,loc}}^{k+1,p} \subset W_{\text{loc}}^{\ell,q}$ holds, then for all $F \in W_{\text{space,loc}}^{\ell,q}(\mathcal{U})$ we have*

$$F \circ \Psi \in W_{\text{space,loc}}^{\ell,q}(\Omega).$$

PROOF: The proof is a repetition of that of Lemma A.2 in [4]. \square

LEMMA A.2 *Let $0 \leq m \leq \ell \leq k$, $q, p \in [1, \infty]$, $kp > n$. Suppose that (ℓ, q) is such that the n -dimensional Sobolev embedding $W_{\text{space,loc}}^{k,p} \subset W_{\text{space,loc}}^{\ell,q}$ holds. Then the product map*

$$W_{\text{space,loc}}^{k-m,p} \times W_{\text{space,loc}}^{\ell,q} \ni (f, g) \longrightarrow fg \in W_{\text{space,loc}}^{\ell-m,q}$$

is continuous.

PROOF: The proof is a repetition of that of Lemma A.4 in [4]. \square

Consider, thus, a connected paracompact Hausdorff $n + 1$ -dimensional manifold \mathcal{M} of C^1 differentiability class. We shall say that \mathcal{M} is of $W_{\text{space,loc}}^{k+1,p}$ differentiability class if \mathcal{M} has an atlas for which all the transition functions are of $W_{\text{space,loc}}^{k+1,p}$ differentiability class. Unless indicated otherwise, the Lebesgue measure in local coordinates is used. The differentiability index is assumed to be integer throughout this Appendix.

Consider functions or tensor fields which, in some local coordinate system, are of differentiability class, $W_{\text{space,loc}}^{\ell,q}$. This class of tensors will be said to be *intrinsically defined* if the differentiability class is preserved under $W_{\text{space,loc}}^{k+1,p}$ coordinate transformations. We have the following

PROPOSITION A.3 *Let (\mathcal{M}, g) be a $W_{\text{space,loc}}^{k+1,p}$ manifold, $kp > n$, $p \in [1, \infty]$.*

1. *If the Sobolev embedding*

$$W_{\text{space,loc}}^{k+1,p} \subset W_{\text{space,loc}}^{\ell,q}$$

holds, then the space of $W_{\text{space,loc}}^{\ell,q}$ scalar fields on \mathcal{M} is intrinsically defined.

2. *If the Sobolev embedding*

$$W_{\text{space,loc}}^{k,p} \subset W_{\text{space,loc}}^{\ell,q}$$

holds, then the space of $W_{\text{space,loc}}^{\ell,q}$ tensor fields on \mathcal{M} is intrinsically defined.

PROOF: This follows from Lemmata A.1-A.2, as in [4, Appendix A] \square

The extension of the above discussion to spinor fields requires the introduction of orthonormal frames, and hence of the metric. Consider, then, a $W_{\text{space,loc}}^{k+1,p}$ manifold \mathcal{M} with a strictly positive definite symmetric two-covariant tensor field g . We shall say that (\mathcal{M}, g) is a *Riemannian $W_{\text{space,loc}}^{k+1,p}$ manifold* if \mathcal{M} is a $W_{\text{space,loc}}^{k+1,p}$ manifold and if g is a Riemannian metric of $W_{\text{space,loc}}^{k,p}$ differentiability class. This is an intrinsically defined notion by Proposition A.3. It is not too difficult to check that:

PROPOSITION A.4 *Let (\mathcal{M}, g) be a $W_{\text{space,loc}}^{k+1,p}$ manifold with a Riemannian metric of $W_{\text{space,loc}}^{k,p}$ differentiability class, $kp > n$, $p \in [1, \infty]$. Then the following hold*

1. *In any coordinate system in the $W_{\text{space,loc}}^{k+1,p}$ atlas the Christoffel coefficients Γ_{jk}^i satisfy*

$$\Gamma_{jk}^i \in W_{\text{space,loc}}^{k-1,p}.$$

2. *The Riemann tensor is of $W_{\text{space,loc}}^{k-2,p}$ differentiability class.*

3. *The curvature scalar $R \equiv g^{ij} R^k_{ikj}$ is of $W_{\text{space,loc}}^{k-2,p}$ differentiability class.*

4. *Assume that $\ell \geq 1$ and suppose that (ℓ, q) is such that the Sobolev embedding $W_{\text{space,loc}}^{k,p} \subset W_{\text{space,loc}}^{\ell,q}$ holds. Let t be a tensor field of $W_{\text{space,loc}}^{\ell,q}$ differentiability class, then for any vector field $X \in W_{\text{space,loc}}^{k,p}$ we have*

$$X^i \nabla_i t \in W_{\text{space,loc}}^{\ell-1,q}.$$

Let (\mathcal{M}, g) be a $W_{\text{space,loc}}^{k+1,p}$ Riemannian manifold and let ON be the bundle of g -orthonormal frames on \mathcal{M} . We can equip ON with a $W_{\text{space,loc}}^{k,p}$ structure by considering only those g -orthonormal sets of vector fields which are all of $W_{\text{space,loc}}^{k,p}$ differentiability class. The set of such (locally defined) frames is not empty: Indeed, on \mathcal{O} , the domain of a coordinate system (x^i) , we can construct a g -orthonormal frame $e_j = e_j^i \partial / \partial x^i$

by performing a Gram–Schmidt orthonormalisation of the basis $\{\partial/\partial x^i\}$. By construction the coordinate coefficients e_j^i of the vector fields e_j are smooth functions of g_{ij} (at least on a neighborhood of the range of values taken by g_{ij}), where the g_{ij} ’s are the coordinate coefficients of the metric g , $g = g_{ij}dx^i dx^j$. Since $kp > n$, the Gagliardo–Moser–Nirenberg inequalities (cf., e.g. [17, Corollaries 6.4.4 and 6.4.5])

$$\forall f, g \in W^{\ell, q} \cap L^\infty \quad \|fg\|_{W^{\ell, q}} \leq C_1(\|f\|_{L^\infty}\|g\|_{W^{\ell, q}} + \|f\|_{W^{\ell, q}}\|g\|_{L^\infty}), \quad (\text{A.3})$$

$$\forall f \in W^{\ell, q} \cap L^\infty \quad \|F(f)\|_{W^{\ell, q}} \leq C_2(\|f\|_{L^\infty})(1 + \|f\|_{W^{\ell, q}}) \quad (\text{A.4})$$

applied to the e_j^i considered as functions of the g_{ij} shows that the vector fields e_j are of $W_{\text{space, loc}}^{k, p}$ differentiability class, as desired.

As in [4], one checks:

PROPOSITION A.5 *Any two (globally or locally defined) g -orthonormal frames of $W_{\text{space, loc}}^{k, p}$ differentiability class are related to each other by a $O(n)$ -rotation of $W_{\text{space, loc}}^{k, p}$ differentiability class.*

PROPOSITION A.6 *Let (\mathcal{M}, g) be a $W_{\text{space, loc}}^{k+1, p}$ spin manifold with a Riemannian metric of $W_{\text{space, loc}}^{k, p}$ differentiability class. Then every spinor bundle carries a natural $W_{\text{space, loc}}^{k, p}$ differentiable structure.*

PROPOSITION A.7 *Let (\mathcal{M}, g) be a $W_{\text{space, loc}}^{k+1, p}$ spin manifold with a Riemannian metric of $W_{\text{space, loc}}^{k, p}$ differentiability class, $kp > n$, $p \in [1, \infty]$. If the Sobolev embedding*

$$W_{\text{space, loc}}^{k, p} \subset W_{\text{space, loc}}^{\ell, q}$$

holds, then the space of $W_{\text{space, loc}}^{\ell, q}$ spinor fields is intrinsically defined.

PROPOSITION A.8 *Let (\mathcal{M}, g) be a $W_{\text{space, loc}}^{k+1, p}$ manifold with a Riemannian metric of $W_{\text{space, loc}}^{k, p}$ differentiability class, $kp > n$, $p \in [1, \infty]$. Then the following hold*

1. *Let e^i be any g -orthonormal frame of $W_{\text{space, loc}}^{k, p}$ differentiability class, then the spin connection coefficients ω_k defined as $\nabla_{e_k}\psi = e_k(\psi) + \omega_k\psi$ satisfy*

$$\omega_k \in W_{\text{space, loc}}^{k-1, p}.$$

2. *If (ℓ, q) is as in Proposition A.7 with $\ell \geq 1$, and if X is a vector field of $W_{\text{space, loc}}^{k, p}$ differentiability class, then ∇_X maps continuously $W_{\text{space, loc}}^{\ell, q}$ spinor fields to $W_{\text{space, loc}}^{\ell-1, q}$ spinor fields:*

$$W_{\text{space, loc}}^{\ell, q} \ni \phi \longrightarrow \nabla_X \phi \in W_{\text{space, loc}}^{\ell-1, q}.$$

In particular the Dirac operator maps continuously $W_{\text{space, loc}}^{\ell, q}$ to $W_{\text{space, loc}}^{\ell-1, q}$.

References

- [1] R.A. Adams, *Sobolev spaces*, Academic Press, N.Y., 1975.
- [2] S. Alinhac and P. Gérard, *Opérateurs pseudo-différentiels et théorème de Nash-Moser*, Savoirs Actuels. [Current Scholarship], InterEditions, Paris, 1991. MR 1172111 (93g:35001)
- [3] T. Aubin, *Nonlinear analysis on manifolds. Monge–Ampère equations*, Springer, New York, Heidelberg, Berlin, 1982.
- [4] R. Bartnik and P.T. Chruściel, *Boundary value problems for Dirac-type equations*, (2003), arXiv:math.DG/0307278.
- [5] Y. Choquet-Bruhat and R. Geroch, *Global aspects of the Cauchy problem in general relativity*, Commun. Math. Phys. **14** (1969), 329–335. MR MR0250640 (40 #3872)
- [6] Y. Choquet-Bruhat and J. York, *The Cauchy problem*, General Relativity (A. Held, ed.), Plenum Press, New York, 1980, pp. 99–172.
- [7] P.T. Chruściel, *On completeness of orbits of Killing vector fields*, Class. Quantum Grav. **10** (1993), 2091–2101, arXiv:gr-qc/9304029.
- [8] ———, *Uniqueness of black holes revisited*, Helv. Phys. Acta **69** (1996), 529–552, Proceedings of Journées Relativistes 96 (N. Straumann, Ph. Jetzer and G. Lavrelashvili, eds), arXiv:gr-qc/9610010.
- [9] ———, *Elements of causality theory*, (2011), arXiv:1110.6706 [gr-qc].
- [10] P.T. Chruściel, G.J. Galloway, and D. Pollack, *Mathematical general relativity: a sampler*, Bull. Amer. Math. Soc. (N.S.) **47** (2010), 567–638, arXiv:1004.1016 [gr-qc]. MR 2721040 (2011j:53152)
- [11] P.T. Chruściel and J. Grant, *On Lorentzian causality with continuous metrics*, (2011), arXiv:1110.0400 [gr-qc].
- [12] P.T. Chruściel and D.B. Singleton, *Nonsmoothness of event horizons of Robinson-Trautman black holes*, Commun. Math. Phys. **147** (1992), 137–162. MR 1171763 (93i:83059)
- [13] B.K. Driver, *Analysis*, http://www.math.ucsd.edu/~bdriver/231-02-03/Lecture_Notes/PDE-Anal-Book/analpde1.pdf.
- [14] A. Fathi and A. Siconolfi, *On smooth time functions*, Math. Proc. Camb. Phil. Soc. **155** (2011), 1–37, Available on CJO doi:10.1017/S0305004111000661.
- [15] H. Federer, *Geometric measure theory*, Springer Verlag, New York, 1969, (Die Grundlehren der mathematischen Wissenschaften, Vol. 153).
- [16] S.W. Hawking and G.F.R. Ellis, *The large scale structure of space-time*, Cambridge University Press, Cambridge, 1973, Cambridge Monographs on Mathematical Physics, No. 1. MR MR0424186 (54 #12154)
- [17] L. Hörmander, *Lectures on nonlinear hyperbolic differential equations*, Math. & Appl. 26, Springer, 1997.
- [18] T.J.R. Hughes, T. Kato, and J.E. Marsden, *Well-posed quasi-linear second-order hyperbolic systems with applications to nonlinear elastodynamics and general relativity*, Arch. Rat. Mech. Anal. **63** (1977), 273–294.

- [19] J.L. Kelley, *General topology*, Springer-Verlag, New York, 1975, Reprint of the 1955 edition [Van Nostrand, Toronto, Ont.], Graduate Texts in Mathematics, No. 27. MR 0370454 (51 #6681)
- [20] S. Klainerman and I. Rodnianski, *Ricci defects of microlocalized Einstein metrics*, Jour. Hyperbolic Diff. Equ. **1** (2004), 85–113. MR MR2052472 (2005f:58048)
- [21] ———, *The causal structure of microlocalized rough Einstein metrics*, Ann. of Math. (2) **161** (2005), 1195–1243. MR MR2180401 (2007d:58052)
- [22] ———, *Rough solutions of the Einstein-vacuum equations*, Ann. of Math. (2) **161** (2005), 1143–1193. MR MR2180400 (2007d:58051)
- [23] P. Mattila, *Geometry of sets and measures in Euclidean spaces*, Cambridge Studies in Advanced Mathematics, vol. 44, Cambridge University Press, Cambridge, 1995, Fractals and rectifiability. MR 1333890 (96h:28006)
- [24] D. Maxwell, *Rough solutions of the Einstein constraint equations*, J. Reine Angew. Math. **590** (2006), 1–29, arXiv:gr-qc/0405088. MR MR2208126 (2006j:58044)
- [25] G. Métivier, *Para-differential calculus and applications to the Cauchy problem for nonlinear systems*, Centro di Ricerca Matematica Ennio De Giorgi (CRM) Series, vol. 5, Edizioni della Normale, Pisa, 2008, <http://www.math.u-bordeaux1.fr/~metivier/Total.Mai08v2.pdf>. MR 2418072 (2010d:35430)
- [26] D. Parlongue, *Geometric uniqueness for non-vacuum Einstein equations and applications*, arXiv:1109.0644 [math-ph].
- [27] F. Planchon and I. Rodnianski, *On uniqueness for the Cauchy problem in general relativity*, (2007), in preparation.
- [28] H. Ringström, *The Cauchy problem in general relativity*, ESI Lectures in Mathematics and Physics, European Mathematical Society (EMS), Zürich, 2009. MR 2527641 (2010j:83001)
- [29] H. Whitney, *Differentiable manifolds*, Ann. of Math. (2) **37** (1936), 645–680. MR 1503303